

# On the diameter of Kronecker graphs

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## Abstract

It is shown that a.a.s. as soon as a Kronecker graph becomes connected its diameter is bounded by a constant.

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## 1. Introduction

A Kronecker graph is a random graph with vertex set  $V = \{0, 1\}^n$ , where the probability that two vertices  $u, v \in V$  are adjacent strongly depends on the structure of the vectors  $u = (u_1, \dots, u_n)$ , and  $v = (v_1, \dots, v_n)$ . More specifically, let  $\mathbf{P}$  be a symmetric matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 1 & 0 \end{matrix} \\ \begin{matrix} 1 \\ 0 \end{matrix} & \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \end{matrix},$$

where zeros and ones are labels of rows and columns of  $\mathbf{P}$ ,  $\alpha, \beta, \gamma \in [0, 1]$ , and  $\alpha \geq \gamma$ . In the Kronecker graph  $\mathcal{K}(n, \mathbf{P})$  two vertices  $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in V = \{0, 1\}^n$  are adjacent with probability

$$p_{u,v} = \prod_{i=1}^n \mathbf{P}[u_i, v_i],$$

independently for each such pair.

Kronecker graphs were introduced by Leskovec, Chakrabarti, Kleinberg and Faloutsos in [1] to model some real world networks (see also [2], [3], [4]). Since then they have been studied by several authors but their properties are still far from being well understood (see [5] and references therein). In particular, Radcliffe and Young [6] determined the exact threshold for the property that  $\mathcal{K}(n, \mathbf{P})$  is connected, supplementing a slightly weaker result of Mahdian and Xu [7].

**Theorem 1.**

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{K}(n, \mathbf{P}) \text{ is connected}) = \begin{cases} 0 & \text{if } \beta + \gamma = 1, \beta \neq 1 \\ 0 & \text{if } \beta = 1, \alpha = \gamma = 0 \\ 1 & \text{if } \beta = 1, \alpha > 0 \text{ and } \gamma = 0 \\ 1 & \text{if } \beta + \gamma > 1. \end{cases}$$

The main result of this work states that as soon as  $\mathcal{K}(n, \mathbf{P})$  becomes connected its diameter is bounded by a constant.

**Theorem 2.** *If either  $\beta + \gamma > 1$ , or  $\beta = 1$ ,  $\alpha > 0$  and  $\gamma = 0$ , then there exists a constant  $a = a(\alpha, \beta, \gamma)$  such that a.a.s.  $\text{diam}(\mathcal{K}(n, \mathbf{P})) \leq a$ .*

## 2. The idea of the proof

In order to sketch our argument let us recall how one shows that the diameter is bounded from above for the binomial model of random graph  $G(N, p)$ , and for many other random graph models. Typically, since random graphs are good expanders, it is proven first that for some small  $k$  the  $k$ -neighbourhood of each vertex is much larger than  $\sqrt{N}$ . Then, in the second part of the proof, one argues that since two random subsets of vertices of size larger than  $\sqrt{N}$  intersect with large probability, each pair of vertices is a.a.s. connected by a path of length at most  $2k$ . In [7], the diameter of  $\mathcal{K}(n, \mathbf{P})$  was examined under condition  $\gamma < \beta < \alpha$ . This case was solved using the standard results for binomial random graphs. However, for other cases this procedure fails completely. The main reason is that most neighbours of a given vertex  $v$  have a similar structure, and so the events ‘ $x \sim v$ ’ and ‘ $y \sim v$ ’ are strongly correlated. Thus, the  $k$ -neighbourhood of a given vertex is very far from being a random subset, which is crucial for the second step of the procedure. Even more importantly, we do not understand expanding properties of  $\mathcal{K}(n, \mathbf{P})$  and it is hard to control how fast the  $k$ -neighbourhoods of a vertex  $\mathcal{K}(n, \mathbf{P})$  grows, which in most of the other random graph models is quite easy to investigate.

Thus, we apply a different approach. We consider two vertices,  $v$  and  $u$  which are very similar to each other (more specifically, we choose both of them from the middle layer of the  $n$ -cube and assume that they are at small Hamming distance from each other). Then we generate their neighbourhoods at the same time until, for some  $k$ , we observe that the  $k$ -neighbourhood of  $v$  would not expand on an expected rate. This is because many, in fact

most, candidates for  $(k + 1)$ -neighbours of  $v$  has already been placed in the  $i$ -neighbourhood of  $v$  for some  $i \leq k$ . However, the chance that a vertex  $x$  is in the  $i$ -neighbourhood of  $v$  is roughly the same as the probability that  $x$  is in the  $i$ -neighbourhood of  $u$  so, if most potential  $(k + 1)$ -neighbours of  $v$  are already in its  $k$ -th neighbourhood, many of them are also in the  $k$ -th neighbourhood of  $u$ . Consequently, there is a path of length at most  $2k$  joining  $v$  and  $u$ .

The structure of the paper is the following. First we treat a special case  $\beta = 1$ . Then we present the crucial part of our argument showing that the subgraph induced in  $\mathcal{K}(n, \mathbf{P})$  by its middle layer has a.a.s. a small diameter. Finally, we complete the proof showing that a.a.s. each vertex of  $\mathcal{K}(n, \mathbf{P})$  is connected to the middle layer by a short path.

### 3. Case $\beta = 1$

In this section we show that if  $\beta = 1$ ,  $\alpha > 0$ , and  $\gamma = 0$ , then the diameter of  $\mathcal{K}(n, \mathbf{P})$  is a.a.s. bounded by a constant. This set of parameters  $\alpha, \beta, \gamma$  is somewhat special as it is the only case, when  $\gamma + \beta = 1$  and still  $\mathcal{K}(n, \mathbf{P})$  is a.a.s. connected.

We introduce some notation, we shall use throughout the paper. By  $d(v, u)$  we denote a Hamming distance between two vertices  $v$  and  $u$  and  $w(v)$  stands for the weight of a vertex  $v = (v_1, \dots, v_n)$ , i.e. the number of ones in its label

$$w(v) = \sum_{i=1}^n v_i.$$

For a vertex  $v = (v_1, \dots, v_n)$ , we set  $\bar{v} = (1 - v_1, \dots, 1 - v_n)$ .

Now let us go back to the case  $\beta = 1$ . Note that

$$\mathbb{P}(v \sim \bar{v}) = \beta^n = 1,$$

and observe that either  $v$  or  $\bar{v}$  has weight at least  $n/2$ . Thus, to show the assertion it is enough to verify that a.a.s. there exists a path of bounded length between every pair of vertices in  $R$  defined as

$$R = \{v \in V : w(v) \geq n/2\}.$$

Let  $v \in R$  be a vertex of weight  $n/2 \leq w(v) < n$ . We show that it is joined to the vertex  $(1, 1, \dots, 1)$  by a short path. To this end let  $\eta \in (0, 1)$

denote the largest solution of the equality

$$\alpha^\eta(\alpha^2 + 1)^{1/2} = 1 + \eta.$$

Let  $u \in R$  be such that  $w(u) = w(v) + t$  and  $d(v, u) = t$ , where  $0 < t < \eta n$ . Consider a vertex  $x$  such that if  $v_i = 0$ , then  $x_i = 1$ , and  $w(x) = n - w(v) + j$ , for some  $j \in [w(v) - 1]$ . Then

$$\mathbb{P}(x \sim v, x \sim u) = \alpha^j \beta^{n-j} \alpha^{j+t} \beta^{n-j-t} = \alpha^{2j+t}.$$

Hence the probability  $\rho(v, u)$  that  $v$  and  $u$  have no common neighbours in  $\mathcal{K}(n, \mathbf{P})$  is bounded from above by

$$\begin{aligned} \rho(v, u) &\leq \prod_{j=1}^{w(v)-1} (1 - \alpha^{2j+t})^{\binom{w(v)}{j}} \leq \exp \left( - \sum_{j=1}^{w(v)-1} \binom{w(v)}{j} \alpha^{2j+t} \right) \\ &\leq \exp \left( - \alpha^t ((\alpha^2 + 1)^{w(v)} - \alpha^{2w(v)} - 1) \right) \\ &\leq \exp \left( - \frac{(\alpha^\eta (\alpha^2 + 1)^{1/2})^n}{2} \right) = \exp \left( - \frac{1}{2} (1 + \eta)^n \right) = o(|R|^2), \end{aligned}$$

Thus a.a.s. each such pair  $u, v \in R$  has a common neighbour. Therefore every vertex in  $\mathcal{K}(n, \mathbf{P})$  is a.a.s. connected to a vertex  $(1, \dots, 1)$  by a path of length at most  $1/2\eta$ . Consequently, a.a.s. between each pair of vertices of  $\mathcal{K}(n, \mathbf{P})$  there exists a path of length at most  $1/\eta + 2$ , where, let us recall,  $\eta$  is a constant which depends only on  $\alpha$ .

#### 4. The middle layer

In this section we deal with the case when  $\beta + \gamma > 1$ . Let us recall that we always assume that  $\alpha \geq \gamma$  however, since if we decrease  $\alpha$  the diameter can only increase, we may and shall assume that  $\alpha = \gamma$ . Furthermore, for technical reasons, we assume also that  $n$  is even; if this is not the case one can split  $\mathcal{K}(n, \mathbf{P})$  into two disjoint random subgraphs for which the underlying cube is of even dimension, use the result to infer that each of them is a.a.s. of bounded diameter and finally verify that a.a.s. there exists at least one edge joining two of them.

Denote by  $\mathcal{H}$  a subgraph of  $\mathcal{K}(n, \mathbf{P})$  induced by the middle layer, i.e. by the set of vertices

$$U = \{v \in V(\mathcal{K}(n, \mathbf{P})) : w(v) = n/2\}.$$

In this section we show that the diameter of  $\mathcal{H}$  is bounded and the following theorem holds.

**Theorem 3.** *For every  $\alpha = \gamma$  and  $\beta + \gamma > 1$ , there exists a constant  $c$ , such that a.a.s.*

$$\text{diam}(\mathcal{H}) < c.$$

In order to verify Theorem 3 we show that a.a.s. each pair of vertices  $v$  and  $u$  such that  $d(v, u)$  is small is connected in average by at least  $n^2/4$  edge-disjoint paths of bounded length. However, although the expected number of short paths between  $v$  and  $u$  grows exponentially it is hard to control directly the size of the largest family of edge-disjoint such paths. Thus, we use the following simple and natural approach. Each edge of  $\mathcal{H}$  we label randomly and independently with one of  $n^2$  labels, i.e. we split  $\mathcal{H}$  randomly into  $n^2$  edge disjoint parts. Of course each of  $n^2$  parts is the same random object, which can be obtained by keeping edges of  $\mathcal{H}$  with probability  $n^{-2}$ . We denote it by  $\overline{\mathcal{H}}$ . The crucial part of our argument is captured by the following lemma.

**Lemma 4.** *For every  $\alpha = \gamma$  and  $\beta + \gamma > 1$ , there exist constants  $\epsilon > 0$  and  $\xi > 1$  such that for each pair  $v, u$  of vertices of  $\overline{\mathcal{H}}$  such that  $d(u, v) < \epsilon n$  the probability that  $u$  and  $v$  are connected in  $\overline{\mathcal{H}}$  by a path of length at most  $4\lceil \log_\xi 2 \rceil$  is at least  $1/4$ .*

*Proof.* Let  $u, v$  be two vertices from  $U$  such that  $d(u, v) < \epsilon n$ , where  $\epsilon > 0$  is to be chosen later on. We show that  $u$  and  $v$  are connected by a short path in a subgraph  $\widehat{\mathcal{H}}$  of  $\overline{\mathcal{H}}$  which is defined as follows.

Let  $I$  be the set of those  $i \in [n]$  for which  $u_i \neq v_i$ . Clearly  $|I| = d(u, v)$  and, since  $w(u) = w(v) = n/2$ ,  $|I|$  is even. The vertex set of  $\widehat{\mathcal{H}}$  consists only of those vertices which have precisely  $|I|/2$  ones inside  $I$  and, consequently, exactly  $(n - |I|)/2$  ones in  $[n] \setminus I$ . We denote it by  $U_I$ , i.e.

$$U_I = \{x \in U : |\{i \in I : x_i = 1\}| = |\{i \in I : x_i = 0\}| = |I|/2\}.$$

Note that clearly  $u, v \in U_I$ .

Now we delete from  $\overline{\mathcal{H}}$  some edges. So firstly, we leave in  $\widehat{\mathcal{H}}$  only those edges which fulfill the following condition:

$$|\{i : x_i = y_i\} \setminus I| = \frac{\alpha}{\alpha + \beta}(n - |I|). \quad (1)$$

Furthermore, when computing the probability that vertices  $x, y$  for which the above holds are adjacent we would like to ‘ignore’ the part of  $x$  and  $y$  which belong to  $I$ . Note that the probability that such a given pair is connected by an edge in  $\overline{\mathcal{H}}$  is always at least

$$\rho = \rho(\alpha, \beta) = \alpha^{\frac{\alpha}{\alpha+\beta}(n-|I|)} \beta^{\frac{\beta}{\alpha+\beta}(n-|I|)} (\min\{\alpha, \beta\})^{|I|} n^{-2}. \quad (2)$$

Now, each edge of  $\overline{\mathcal{H}}$  for which the probability of existence  $\rho'$  is larger than  $\rho$  above we delete with probability  $\rho/\rho'$ . The edges which remain after these procedures are the edges of  $\widehat{\mathcal{H}}$ .

Note that for every vertices  $x$  and  $y$  of  $\widehat{\mathcal{H}}$  which differ only on the set  $I$  (such as  $u$  and  $v$ ) and all other  $z \in U_I$  the probability that  $z$  is adjacent to  $x$  is precisely the same as the probability that  $z$  is adjacent to  $y$ . Furthermore, all vertices of  $\widehat{\mathcal{H}}$  are, in a way, equivalent. More precisely, for every  $x, y \in U_I$  a natural permutation of coordinates of  $x$  induces a bijection of  $U_I$  onto itself which transforms  $x$  into  $y$  and which preserves measure, i.e. which transform  $\widehat{\mathcal{H}}$  onto itself.

We shall show that degrees of vertices of  $\widehat{\mathcal{H}}$  increase exponentially with  $n$ . Let us first make the following useful observation. It is easy to see (cf. [7]) that the expected size of neighbourhood of a vertex  $v$  in  $\mathcal{K}(n, \mathbf{P})$  is equal to

$$\sum_{r=0}^{w(v)} \sum_{s=0}^{n-w(v)} \binom{w(v)}{r} \binom{n-w(v)}{s} \alpha^r \beta^{w(v)-r} \alpha^s \beta^{n-w(v)-s} = (\alpha + \beta)^n.$$

Thus, the largest term in the sums on the left hand side is at least as large as  $(\alpha + \beta)^n/n^2$ . Since we shall often refer to this fact let us state it explicitly. For every natural numbers  $n, w$  and positive constants  $\alpha, \beta < 1$  we have

$$\binom{w}{\frac{\alpha}{\alpha+\beta}w} \binom{n-w}{\frac{\alpha}{\alpha+\beta}(n-w)} \alpha^{\frac{\alpha}{\alpha+\beta}w} \beta^{\frac{\alpha}{\alpha+\beta}(n-w)} \geq (\alpha + \beta)^n/n^2. \quad (3)$$

Using the above inequality we can easily estimate degrees of vertices in  $\widehat{\mathcal{H}}$ . Let us denote by  $\widehat{N}(x)$  the set of all neighbours of  $x$  in  $\widehat{\mathcal{H}}$ , and, more generally, by  $\widehat{N}_i(x)$  we denote the  $i$ -th neighbourhood of  $x$  (i.e. the set of all  $y$  which lies at the distance  $i$  from  $x$  in  $\widehat{\mathcal{H}}$ ). Then the following holds.

**Fact 5.** *Let  $\epsilon > 0$  be a small constant and let  $I$  be a given subset of  $[n]$  such that  $|I| \leq \epsilon n$ . Then, if  $\epsilon > 0$  is small enough, there exists a constant  $\xi > 1$*

such that for every  $x \in U_I$  we have

$$\mathbb{P}\left(|\widehat{N}(x)| \geq \xi^n\right) = 1 - o(1).$$

*Proof.* For a given  $x \in U_I$  the random variable  $\widehat{N}(x)$  has the binomial distribution  $B(M, \rho)$ , where

$$M = \left( \frac{(n - |I|)/2}{\frac{\alpha}{\alpha + \beta}(n - |I|)/2} \right)^2,$$

and  $\rho$  is given by (2). From (3) and the fact that  $|I| < \epsilon n$  we get

$$\begin{aligned} \mathbb{E}\widehat{N}(x) &= (\min\{\alpha, \beta\})^{|I|} \left( \frac{(n - |I|)/2}{\frac{\alpha}{\alpha + \beta}(n - |I|)/2} \right)^2 \alpha^{\frac{\alpha}{\alpha + \beta}(n - |I|)} \beta^{\frac{\beta}{\alpha + \beta}(n - |I|)} n^{-2} \\ &\geq (\min\{\alpha, \beta\})^{\epsilon n} (\alpha + \beta)^{(1 - \epsilon)n} / n^4. \end{aligned}$$

Hence, if  $\epsilon$  is small enough, for some constant  $\xi > 1$  we have  $\mathbb{E}\widehat{N}(x) \geq 2\xi^n$  and the assertion follows from either Chernoff's or Chebyshev's inequalities.  $\square$

Now let  $k = 2\lceil \log_\xi 2 \rceil$ . Recall that we need to show that

$$\mathbb{P}\left(\widehat{N}_k(v) \cap \widehat{N}_k(u) \neq \emptyset\right) \geq 1/4. \quad (4)$$

In fact we shall show a slightly stronger inequality. Let us first split the set  $\widehat{N}(v)$  randomly into two sets  $\widehat{N}_{(H)}(v)$  and  $\widehat{N}_{(T)}(v)$  by tossing for each vertex of  $\widehat{N}(v)$  a symmetric coin. In the same way we partition  $\widehat{N}(u)$  into sets  $\widehat{N}_{(H)}(u)$  and  $\widehat{N}_{(T)}(u)$ . Furthermore, let  $\widehat{N}_k^{-y}(x)$  denote the  $k$ -neighbourhood of  $x$  in  $\widehat{\mathcal{H}}$  from which we removed an edge  $\{x, y\}$  if such an edge exists. We shall show that

$$\mathbb{P}\left(\bigcup_{x \in \widehat{N}_{(T)}(v)} \widehat{N}_{k-1}^{-v}(x) \cap \bigcup_{y \in \widehat{N}_{(H)}(u)} \widehat{N}_{k-1}^{-u}(y) \neq \emptyset\right) \geq 1/4. \quad (5)$$

Let us assume that (5) does not hold, i.e. that we have

$$\mathbb{P} \left( \bigcup_{x \in \hat{N}_{(T)}(v)} \hat{N}_{k-1}^{-v}(x) \cap \bigcup_{y \in \hat{N}_{(H)}(u)} \hat{N}_{k-1}^{-u}(y) \neq \emptyset \right) < 1/4. \quad (6)$$

We shall show that (6) leads to a contradiction.

Let us start with a simple but crucial observation. In order to check if the event

$$\bigcup_{x \in \hat{N}_{(T)}(v)} \hat{N}_{k-1}^{-v}(x) \cap \bigcup_{y \in \hat{N}_{(H)}(u)} \hat{N}_{k-1}^{-u}(y) \neq \emptyset$$

holds we need first to generate the set  $\bigcup_{x \in \hat{N}_{(T)}(v)} \hat{N}_{k-1}^{-v}(x)$  and then generate vertices of  $\bigcup_{y \in \hat{N}_{(H)}(u)} \hat{N}_{k-1}^{-u}(y)$  until the moment when we meet the first vertex which belongs to both of these sets. But the distribution of  $\bigcup_{y \in \hat{N}_{(H)}(u)} \hat{N}_{k-1}^{-u}(y)$  is identical with  $\bigcup_{y \in \hat{N}_{(H)}(v)} \hat{N}_{k-1}^{-v}(y)$ . Thus,

$$\begin{aligned} \mathbb{P} \left( \bigcup_{x \in \hat{N}_{(T)}(v)} \hat{N}_{k-1}^{-v}(x) \cap \bigcup_{y \in \hat{N}_{(H)}(u)} \hat{N}_{k-1}^{-u}(y) \neq \emptyset \right) \\ \geq \mathbb{P} \left( \bigcup_{x \in \hat{N}_{(T)}(v)} \hat{N}_{k-1}^{-v}(x) \cap \bigcup_{y \in \hat{N}_{(H)}(v)} \hat{N}_{k-1}^{-v}(y) \neq \emptyset \right) \end{aligned}$$

where we have inequality instead of equality because, perhaps,  $\hat{N}_{(T)}(v) \cap \hat{N}_{(H)}(u) \neq \emptyset$ . Hence, from (6) it follows that

$$\mathbb{P} \left( \bigcup_{x \in \hat{N}_{(T)}(v)} \hat{N}_{k-1}^{-v}(x) \cap \bigcup_{y \in \hat{N}_{(H)}(v)} \hat{N}_{k-1}^{-v}(y) \neq \emptyset \right) < 1/4,$$

and, since we may toss a coin after we generate all vertices from  $\hat{N}(v)$ ,

$$\mathbb{P} \left( \exists_{x, y \in \hat{N}(v), x \neq y} : \hat{N}_{k-1}^{-v}(x) \cap \hat{N}_{k-1}^{-v}(y) \neq \emptyset \right) < 1/2. \quad (7)$$

The above inequality seems to lead to contradiction at very first sight. Indeed, since the degree of vertices of  $\hat{\mathcal{H}}$  grows like  $\xi^n$ , after finite number of steps we explore all the vertices of  $\hat{\mathcal{H}}$  and simply there will be no place for



new vertices. However, keeping in mind that the neighbours of a vertex of  $\widehat{\mathcal{H}}$  are strongly correlated, we have to exclude the possibility that each  $\widehat{N}_{k-1}^{-v}(x)$  spans a dense graph yet for different  $x$  and  $y$  the sets  $\widehat{N}_{k-1}^{-v}(x)$  and  $\widehat{N}_{k-1}^{-v}(y)$  are disjoint.

Thus, let  $J$  be a random variable, defined as

$$J = \min\{i : |\widehat{N}_i(v)| \leq \xi^{ni/2}\}.$$

Note that since for  $k = 2\lceil \log_\xi 2 \rceil$  we have  $\xi^{nk/2} > 2^n$ ,  $J$  takes only finite number of values, more precisely, we have  $J \in [k]$ . Let  $j \in [k]$  maximize the value of  $\mathbb{P}(J = j)$ . Then, clearly,

$$\mathbb{P}\left(|\widehat{N}_{j-1}(v)| \geq \xi^{n(j-1)/2} \ \& \ |\widehat{N}_j(v)| \leq \xi^{nj/2}\right) \geq \mathbb{P}(J = j) \geq 1/k. \quad (8)$$

Since with large probability we have  $|\widehat{N}(v)| \geq \xi^n$ , and due to (8) for each  $x \in \widehat{N}(v)$  we have  $|\widehat{N}_{j-1}(x)| \sim |\widehat{N}_{j-1}(v)| \geq \xi^{n(j-1)/2}$ , then, clearly, if most of  $\widehat{N}_{j-1}(x)$ ,  $\widehat{N}_{j-1}(y)$  are disjoint (see (6)) we must also have  $|\widehat{N}_j(v)| \geq \xi^{n(j+1)/2}$  contradicting (8).

Let us make the above heuristic argument rigorous. Recall that with probability  $1 - o(1)$  we have  $|\widehat{N}(v)| \geq \xi^n$  (Fact 5). Furthermore, since for every  $x \in \widehat{N}(v)$  the random variables  $|\widehat{N}_{j-1}(x)|$  and  $|\widehat{N}_{j-1}(v)|$  have identical distribution, and the degree of  $x$  is binomially distributed with exponential expectation, from (8) it follows that for every  $x \in \widehat{N}_1(v)$  we have

$$\mathbb{P}\left(|\widehat{N}_{j-1}^{-v}(x)| \geq \xi^{n(j-1)/2}/2\right) \geq 1/(2k). \quad (9)$$

Let us generate all neighbours of  $v$  and order them in a sequence  $x_1, x_2, \dots, x_r$ , where, as we have just observed, with probability  $1 - o(1)$  we have  $r \geq \xi^n$ . Then, for each  $x_i$  we generate a set  $W_i$  in the following way. We set  $W_1 = \widehat{N}_{j-1}^{-v}(x_1)$ . Once all sets  $W_1, \dots, W_{i-1}$  are found, we generate vertices of the  $(j-1)$ -neighbourhoods of  $x_i$  one by one and stop when we either find the whole neighbourhood  $\widehat{N}_{j-1}^{-v}(x_i)$ , or when we first hit an element of  $\bigcup_{s=1}^{i-1} W_s$ . In this latter case we set as  $W_i$  the set of all vertices of  $\widehat{N}_{j-1}^{-v}(x_i)$  generated so far and mark  $x_i$  as *bad*.

**Fact 6.** *With probability  $1 - o(1)$  none of the vertices  $x_1, x_2, \dots, x_t$ , where  $t = \xi^{2n/3}$ , is bad.*

*Proof.* During the execution of the procedure of generating  $W_1, W_2, \dots, W_r$ , the probabilities  $\theta_i$  that  $W_i$  is bad increases with  $i$ . Thus, for every  $i \in [t]$ ,  $\theta_i \leq \theta_{t+1}$ . On the other hand the probability that there are bad vertices among  $x_{t+1}, x_{t+2}, \dots, x_r$  is bounded from below by  $1 - (1 - \theta_{t+1})^{r-t}$  while, from our assumption (7), it is bounded from above by  $1/2$ . Hence,

$$1 - (1 - \theta_{t+1})^{r-t} < 1/2,$$

and so, very crudely,  $\theta_{t+1} \leq \xi^{-3n/4}$ . Thus, the probability that one of vertices  $x_1, \dots, x_t$  is bad is bounded from above by

$$\sum_{i=1}^t \theta_i \leq t\theta_{t+1} \leq \xi^{2n/3} \xi^{-3n/4} = o(1). \quad \square$$

Since with probability  $1 - o(1)$  sets  $W_i$ ,  $i = 1, 2, \dots, t$ , are disjoint and thus equal to  $\widehat{N}_{j-1}^{-v}(x_i)$ , and since each of such sets with probability  $1/(2k)$  have size at least  $\xi^{(j-1)n/2}/2$ , with probability  $1 - o(1)$  we have

$$|\widehat{N}_j(v)| \geq \left| \bigcup_{i=1}^t \widehat{N}_{j-1}^{-v}(x_i) \right| / (3k) \geq t\xi^{(j-1)n/2} / (6k) > \xi^{jn/2}$$

contradicting (8). Thus, (6) leads to a contradiction and Lemma 4 holds.  $\square$

*Proof of Theorem 3.* Let  $\epsilon$  and  $\xi$  be constants for which the assertion of Lemma 4 holds. Consider two vertices  $v, u$  of  $\mathcal{H}$  such that  $d(u, v) < \epsilon n$  and let  $X_{v,u}$  denote the maximum number of edge-disjoint path of length at most  $4\lceil \log_\xi 2 \rceil$  joining  $v$  and  $u$  in  $\mathcal{H}$ . From Lemma 4 it follows that  $\mathbb{E}X_{v,u} \geq n^2/4$ . Since adding or removing a single edge cannot affect the value of  $X_{v,u}$  by more than one, and the fact that  $X_{v,u} \geq m$  can be verified by exposing only  $4m\lceil \log_\xi 2 \rceil$  edges, from Talagrand's inequality (see, for instance, [8], Theorem 2.29) we get that for some constant  $\eta = \eta(\epsilon, \xi) > 0$

$$\mathbb{P}(X_{v,u} < n^2/8) \leq \exp(-\eta n^2).$$

Thus, by the first moment method a.a.s. each pair of vertices  $v, u$ , of  $\mathcal{H}$  such that  $d(v, u) \leq \epsilon n$  is connected by a path of length at most  $4\lceil \log_\xi 2 \rceil$ . To complete the proof it is enough to observe that for every pair of vertices  $u, u'$  of  $\mathcal{H}$  one can find a sequence

$$u = u_0, u_1, \dots, u_r = u'$$

such that for  $i \in [r]$ ,  $d(u_i, u_{i-1}) < \epsilon n$ , and  $r < 2/\epsilon$ . Consequently,

$$\text{diam}(\mathcal{H}) \leq 8 \lceil \log_\xi 2/\epsilon \rceil. \quad \square$$

## 5. A final touch: reaching the middle layer

In this section we prove that a.a.s. every vertex of  $\mathcal{K}(n, \mathbf{P})$  is joined to the middle layer by a short path, i.e. the following result holds.

**Theorem 7.** *For every  $\alpha = \gamma$  and  $\beta + \gamma > 1$  there exists  $c' = c'(\alpha, \beta)$  such that a.a.s. every vertex  $v$  of  $\mathcal{K}(n, \mathbf{P})$  is connected by a path of length at most  $c'$  to the middle layer of  $\mathcal{K}(n, \mathbf{P})$ .*

Theorem 7 is a direct consequence of the following three lemmata.

**Lemma 8.** *Let  $\alpha = \gamma$  and  $\beta + \gamma > 1$ . Then a.a.s. each vertex  $v$  of  $\mathcal{K}(n, \mathbf{P})$  of weight  $w(v) \neq n/2$  has a neighbour of weight*

$$\frac{n}{2} + \frac{\alpha - \beta}{\alpha + \beta} \left( w(v) - \frac{n}{2} \right).$$

*Proof.* For a vertex  $v$  consider the set of all vertices of  $\mathcal{K}(n, \mathbf{P})$  which have respectively  $\frac{\alpha}{\alpha + \beta} w(v)$  and  $\frac{\alpha}{\alpha + \beta} (n - w(v))$  common ones and common zeros with  $v$ . There are precisely

$$\binom{w(v)}{\frac{\alpha}{\alpha + \beta} w(v)} \binom{n - w(v)}{\frac{\alpha}{\alpha + \beta} (n - w(v))}$$

of them. Each of these vertices has weight

$$\frac{\alpha}{\alpha + \beta} w(v) + \frac{\beta}{\alpha + \beta} (n - w(v)) = \frac{n}{2} + \frac{\alpha - \beta}{\alpha + \beta} \left( w - \frac{n}{2} \right).$$

Moreover, each of them is adjacent to  $v$  with probability

$$\alpha^{\frac{\alpha}{\alpha + \beta} n} \beta^{\frac{\beta}{\alpha + \beta} n}.$$

Let  $X(v)$  be a random variable which counts those neighbours of  $v$ . Then, by (3), we get

$$\mathbb{E}X(v) = \binom{w(v)}{\frac{\alpha}{\alpha + \beta} w(v)} \binom{n - w(v)}{\frac{\alpha}{\alpha + \beta} (n - w(v))} \alpha^{\frac{\alpha}{\alpha + \beta} n} \beta^{\frac{\beta}{\alpha + \beta} n} \geq \frac{(\alpha + \beta)^n}{n^2}.$$

By Chernoff bound, with probability at least

$$1 - \exp(-\mathbb{E}X(v)/8) \geq 1 - \exp(-n^2),$$

$v$  has more than  $\frac{1}{2}\mathbb{E}X(v)$  neighbours of weight

$$\frac{n}{2} + \frac{\alpha - \beta}{\alpha + \beta} \left( w - \frac{n}{2} \right).$$

Hence, by the first moment method, the probability that  $\mathcal{K}(n, \mathbf{P})$  contains a vertex which does not have this property tends to 0 as  $n \rightarrow \infty$ .  $\square$

**Lemma 9.** *Let  $\alpha = \gamma$  and  $\beta + \gamma > 1$ ,  $\epsilon > 0$ , and let*

$$b = \log_{\left| \frac{\alpha - \beta}{\alpha + \beta} \right|}(\epsilon).$$

*Then a.a.s. every vertex  $v$  of  $\mathcal{K}(n, \mathbf{P})$  such that  $|w(v) - n/2| > \epsilon n$  is connected by a path of length at most  $b$  to a vertex  $u$  with weight  $w(u)$  such that*

$$\left| w(u) - \frac{n}{2} \right| \leq \epsilon \left( w(v) - \frac{n}{2} \right).$$

*Proof.* By Lemma 8, a.a.s. for each vertex  $v$  there exists a path

$$v = v_0 \sim v_1 \sim v_2 \sim \dots v_b$$

such that

$$w(v_i) = \frac{n}{2} + \frac{\alpha - \beta}{\alpha + \beta} \left( w(v_{i-1}) - \frac{n}{2} \right),$$

for  $i \in [b]$ . Solving the above recurrence we get

$$w(v_b) = \frac{n}{2} + \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^b \left( w(v) - \frac{n}{2} \right),$$

and so the assertion follows.  $\square$

**Lemma 10.** *Let  $\alpha = \gamma$  and  $\beta + \gamma > 1$ . Moreover, let  $\epsilon > 0$  be such that*

$$(\alpha + \beta)^{1-\epsilon} (4\alpha\beta)^\epsilon > 1. \tag{10}$$

*Then a.a.s. every vertex  $v$  of  $\mathcal{K}(n, \mathbf{P})$  of weight  $w(v)$  such that  $|w(v) - n/2| \leq \epsilon n/2$  has a neighbour with weight  $n/2$ .*

*Proof.* For a given vertex  $v$ , let  $|w(v) - n/2| = \eta n/2$ , where  $\eta \leq \epsilon$ . Let us choose  $(1 - \eta)n/2$  ones and  $(1 - \eta)n/2$  zeros in the label of  $v$ . Let  $A$  denote the set of those vertices, which have precisely  $\frac{\alpha}{\alpha+\beta}(1 - \eta)n/2$  ones among the chosen one-positions in the label of  $v$ , a  $\frac{\alpha}{\alpha+\beta}(1 - \eta)n/2$  zeros among the chosen zero-positions in the label of  $v$ , and have  $\eta n/2$  zeros among remaining  $\eta n$  positions. Then

$$|A| = \binom{(1 - \eta)n/2}{\frac{\alpha}{\alpha+\beta}(1 - \eta)n/2}^2 \binom{\eta n}{\frac{1}{2}\eta n}.$$

Furthermore, for every  $u \in A$ , the probability that  $u$  is a neighbour of  $v$  is

$$\alpha^{\frac{\alpha}{\alpha+\beta}(1-\eta)n} \beta^{\frac{\beta}{\alpha+\beta}(1-\eta)n} \alpha^{\frac{1}{2}\eta} \beta^{\frac{1}{2}\eta}.$$

Let  $X(v)$  be a random variable which counts neighbours of  $v$  in  $A$ . Then, by (3), we have

$$\begin{aligned} \mathbb{E}X(v) &= \binom{(1 - \eta)n/2}{\frac{\alpha}{\alpha+\beta}(1 - \eta)n/2}^2 \binom{\eta n}{\frac{1}{2}\eta n} \alpha^{\frac{\alpha}{\alpha+\beta}(1-\eta)n} \beta^{\frac{\beta}{\alpha+\beta}(1-\eta)n} \alpha^{\frac{1}{2}\eta} \beta^{\frac{1}{2}\eta} \\ &\geq ((\alpha + \beta)^{1-\eta} (4\alpha\beta)^\eta)^n / n^2. \end{aligned}$$

As  $\eta \leq \epsilon$ , from (10) we get

$$\mathbb{E}X(v) \geq \zeta^n,$$

for some  $\zeta > 1$ . Moreover for every  $u \in A$ ,

$$w(u) = \frac{\alpha}{\alpha + \beta}(1 - \eta)n/2 + \frac{\beta}{\alpha + \beta}(1 - \eta)n/2 + \eta n/2 = n/2.$$

By Chernoff's inequality, with probability at least  $1 - \exp(-\mathbb{E}X(v)/8)$ ,  $v$  has a neighbour with weight  $n/2$ . Thus, the probability that some vertex  $v$  of weight  $w(v)$  such that

$$|w(v) - n/2| \leq \epsilon n/2,$$

has no neighbours of weight  $n/2$  tends to 0 as  $n \rightarrow \infty$ .  $\square$

*Proof of Theorem 7.* Theorem 7 is a straightforward consequence of Lemmata 8, 9, and 10.  $\square$

Now it is easy to complete the proof of Theorem 2.

*Proof of Theorem 2.* From Theorem 7 we know that for some constant  $c'$  a.a.s. from every vertex  $v$  of  $\mathcal{K}(n, \mathbf{P})$  we can reach the middle layer in at most  $c'$  steps. Moreover, Theorem 3 states that for some constant  $c$  a.a.s. the diameter of the subgraph of  $\mathcal{K}(n, \mathbf{P})$  induced by the middle layer is at most  $c$ . Consequently, the diameter of  $\mathcal{K}(n, \mathbf{P})$  is a.a.s. bounded from above by  $c + 2c'$ .  $\square$

**Acknowledgment:** The authors were partially supported by Maestro NCN grant 2012/06/A/ST1/00261.

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